Makarov's Thm: Lower bound

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Thm. $\dim \omega \geq 1$. Moreover, $\exists C>0$: $h(t):= t\exp\left(C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}\right)$ such that $K\subset\partial\Omega$, $\omega(K)>0 \Rightarrow m_h(K)>0$.

("Moreover", since $\forall \alpha<1$, $\lim \frac{h(t)}{t^\alpha}=0$).

Lemma (Rohde) Let $0<\delta<\varepsilon$, $\frac{1}{2}\leq r<1$, $A\subset\mathbb{T}$.

If 1) $|\varphi(A)|\leq\varepsilon$.

2) $\forall \zeta\in A$: $|\varphi(r\zeta)-\varphi(\zeta)|\leq\varepsilon$

3) $(1-r)|\varphi'(r\zeta)|\geq\delta$ $\forall\zeta\in A$.

Then A can be covered by $\leq C_1\left(\frac{\varepsilon}{\delta}\right)^2$ sets of diameter $\leq 1-r$.

Pf Take small c. $\#$ dyadic squares of size $c\delta$, let $(Q_k)_{k=1}^m=\{$dyadic squares $:\varphi(rA)\cap Q_k\}\neq\emptyset$.

By 1)+2), $|\varphi(rA)|\leq 3\varepsilon$. So

$\text{Area}(Q_1\cup Q_2\ldots\cup Q_m)=(c\delta)^2 m\leq C_2\varepsilon^2$, so $m\leq C_1\left(\frac{\varepsilon}{\delta}\right)^2$.

Let $A_k:=\{\zeta\in A: f(r\zeta)\in Q_k\}$. Check that $|A_k|\leq 1-r$.

Assume $|A_k|>1-r$: $\exists\zeta,\zeta'\in A_k$: $|r\zeta-r\zeta'|>1-r$ $\Rightarrow$

(by 3) $\delta<(1-r^2)|\varphi'(r\zeta)|\overset{\text{distortion}}{\leq} C_5|f(r\zeta)-f(r\zeta')|\leq 2C_5|Q_k|<2c_5c\delta$. Take now $c<\frac{1}{2c_5}$ to get contradiction ∎

Pf of lower bound:

Reminder Thm (Makarov's LIL): $\exists c>0$ - absolute:

$$\text{A.e. } \zeta\in\mathbb{T}: \varlimsup_{r\to 1-}\frac{|\log f'(r\zeta)|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}}\leq c$$

Find $A'\subset\varphi^{-1}(K)$ with $m_1(A')>0$.

and $\left|\log|\varphi'(r\zeta)|\right|\leq\psi(r):=C\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}$ for $\forall r>r_0$ and $\forall\zeta\in A'$.

Easy to see, by integration, that $|\varphi(\zeta) - \varphi(r\zeta)| = C(1-r)e^{\cdots}$.

Let B_k – open cover of $\varphi(A') \subset K$.

$A_k := \varphi^{-1}(B_k)$, $\quad \varepsilon_k := |B_k|$.

Define r_k by $\varepsilon_k = (1-r_k)\exp(\psi(r_k))$.

$\delta_k := (1-r_k)\exp(-\psi(r_k))$.

By Rohde's Lemma, A_k can be covered by $\leq C\left(\frac{\varepsilon_k}{\delta_k}\right)^2$ sets of diameter $(1-r_k)$. So

$$m_1(A') \leq \sum m_1(A_k) \lesssim \sum (1-r_k)\exp(4\psi(r_k)) \lesssim \sum h(\varepsilon_k).$$

So $m_h(K) \geq m_h(\varphi(A')) \gtrsim m_1(A') > 0$ ■